Mohr's Circle Analysis Using Linear Algebra and Numerical Methods

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Abstract

Mohr's Circle–or more generally the stress equilibrium in solids–is a well known method to analyze the stress state of a two- or three-dimensional solid. Most engineers are exposed to its derivation using ordinary algebra, especially as it relates to the determination of principal stresses and invariants for comparison with failure criteria. In this presentation, an approach using linear algebra will be set forth, which is interesting not only from the standpoint of the stress state but from linear algebra considerations, as the problem makes an excellent illustration of linear algebra concepts from a real geometric system. The concept of a numerical solution to this problem will also be discussed.

1. Introduction

The analysis of the stress state of a solid at a given point is a basic part of mechanics of materials. Although such analysis is generally associated with the theory of elasticity, in fact it is based on static equilibrium, and is also valid for the plastic region as well. In this way it is used in non-linear finite element analysis, among other applications. The usual objective of such an analysis is to determine the principal stresses at a point, which in turn can be compared to failure criteria to determine local deformation.

In this approach the governing equations will be cast in a linear algebra form and the problem solved in this way, as opposed to other types of solutions given in various textbooks. Doing it in this way can have three results:

1. It allows the abstract concepts of linear algebra to be illustrated well in a physical problem.
2. It allows the physics of the determination of principal stresses to be seen in a different way with the mathematics.
3. It opens up the problem to numerical solution, as opposed to the complicated closed-form solutions usually encountered, of the invariants, principal stresses or direction cosines.

2. Two-Dimensional Analysis

2.1. Eigenvalues, Eigenvectors and Principal Stresses

The simplest way to illustrate this is to use two-dimensional analysis. Even with this simplest case, the algebra can become very difficult very quickly, and the concepts themselves obscured. Consider first the stress state shown in Figure 1, with the notation which will be used in the rest of the article.

The theory behind this figure is described in many mechanics of materials textbooks; for this case the presentation of Jaeger and Cook [4] is used. The element is in static equilibrium along both axes. In order for the summation of moments to be zero,

$$\tau_{xy} = \tau_{yx} \tag{1}$$

The angles are done a little differently than usual; this is to allow an easier transition when three-dimensional analysis is considered. The direction cosines based on these angles are as follows:

$$l = cos\alpha \tag{2}$$

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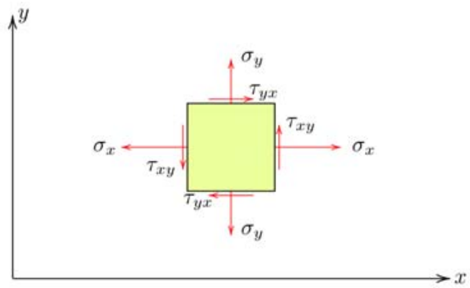

Figure A.1: Stress in two dimensions.

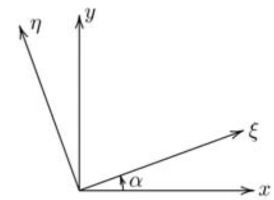

Figure A.2: Rotation of axes.

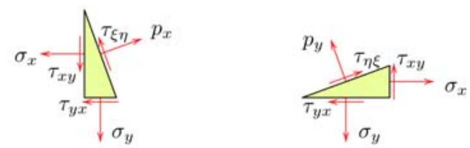

Figure A.3: Stresses in a rotated system of coordinates.

Figure 1: Stress State Coordinates (modified from Verruijt and van Bars [8])

$$m = cos\beta \tag{3}$$

Now consider the components p_x, p_y of the stress vector p on their respective axes. Putting these into matrix form, they are computed as follows:

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} \begin{bmatrix} l \\ m \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \tag{4}$$

which is a classic $Ax = b$ type of problem. At this point there are some things about the matrix in Equation 4 that need to be noted (DeRusso et al. [2]):

1. It is square.
2. It is real and symmetric. Because of these two properties:
   (a) The eigenvalues (and thus the principal stresses, as will be shown) are real. Since for two-dimensional space the equations for the principal stresses are quadratic, this is not a "given" from pure algebra alone.
   (b) The eigenvectors form an orthogonal set, which is important in the diagonalization process.
3. The sum of the diagonal entries of the matrix, referred to as the trace, is equal to the sum of the values of the eigenvalues. As will be seen, this means that, as we rotate the coordinate axes, the trace remains invariant.

At this point there are two related questions that need to be asked. The first is whether static equilibrium will hold if the coordinate axes are rotated. The obvious answer is "yes," otherwise there would be no real static equilibrium. The stresses will obviously change in the process of rotation. These values can be found using a rotation matrix and multiplying the original matrix as follows (Strang [7]):

$$\begin{bmatrix} cos\alpha & -sin\alpha \\ sin\alpha & cos\alpha \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \sigma'_x & \tau'_{xy} \\ \tau'_{xy} & \sigma'_y \end{bmatrix} \tag{5}$$

The rotation matrix is normally associated with Wallace Givens, who taught at the University of Tennessee at Knoxville. The primed values represent the stresses in the rotated coordinate system. We can rewrite the rotation matrix as follows, to correspond with the notation given above:

$$\begin{bmatrix} l & -m \\ m & l \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix} = \begin{bmatrix} \sigma'_x & \tau'_{xy} \\ \tau'_{xy} & \sigma'_y \end{bmatrix} \tag{6}$$

The second question is this: is there an angle (or set of direction cosines) where the shear stresses would go away, leaving only normal stresses? Because of the properties of the matrix, the answer to this is also "yes," and involves the process of diagonalizing the matrix. The diagonalized matrix (the matrix with only nonzero values along the diagonal) can be found if the eigenvalues λ of the matrix can be found, i.e., if the following equation can be solved for λ:

$$\left[\begin{array}{cc} \sigma_x - \lambda & \tau_{xy} \\ \tau_{xy} & \sigma_y - \lambda \end{array}\right] = 0 \tag{7}$$

To accomplish this, we take the determinant of the left hand side of Equation 7, namely

$$\lambda^2 - (\sigma_x + \sigma_y)\,\lambda - \left({\tau_{xy}}^2 - \sigma_x\sigma_y\right) = 0 \tag{8}$$

If we define

$$J_1 = \sigma_x + \sigma_y \tag{9}$$

and

$$J_2 = {\tau_{xy}}^2 - \sigma_x\sigma_y \tag{10}$$

Equation 8 can be rewritten as

$$\lambda^2 - J_1\lambda - J_2 = 0 \tag{11}$$

The quantities I_1, I_2 are referred to as the invariants; they do not change as the axes are rotated. The first invariant is the trace of the system, which was predicted to be invariant earlier. These are very important, especially in finite element analysis, where failure criteria are frequently computed relative to the invariants and not the principal stresses in any combination. This is discussed at length in Owen and Hinton [6].

This is the characteristic polynomial of the matrix of Equation 7. Most people generally don't associate matrices and polynomials, but every matrix has an associated characteristic polynomial, and conversely a polynomial can have one or more corresponding matrices.

The solution of Equation 8 produces the two eigenvalues of the matrix. The first eigenvalue of the matrix is

$$\lambda_1 = 1/2\,\sigma_y + 1/2\,\sigma_x + 1/2\,\sqrt{{\sigma_y}^2 - 2\,\sigma_x\sigma_y + {\sigma_x}^2 + 4\,{\tau_{xy}}^2} \tag{12}$$

and the second

$$\lambda_1 = 1/2\,\sigma_y + 1/2\,\sigma_x - 1/2\,\sqrt{{\sigma_y}^2 - 2\,\sigma_x\sigma_y + {\sigma_x}^2 + 4\,{\tau_{xy}}^2} \tag{13}$$

We can also determine the eigenvectors from this. Without going into the process of determining these, the first eigenvector is

$$\bar{x}_1 = \left[\begin{array}{c} \frac{-1/2\,\sigma_y + 1/2\,\sigma_x + 1/2\,\sqrt{{\sigma_y}^2 - 2\,\sigma_x\sigma_y + {\sigma_x}^2 + 4\,{\tau_{xy}}^2}}{\tau_{xy}} \\ 1 \end{array}\right] \tag{14}$$

and the second is

$$\bar{x}_2 = \left[\begin{array}{c} \frac{-1/2\,\sigma_y + 1/2\,\sigma_x - 1/2\,\sqrt{{\sigma_y}^2 - 2\,\sigma_x\sigma_y + {\sigma_x}^2 + 4\,{\tau_{xy}}^2}}{\tau_{xy}} \\ 1 \end{array}\right] \tag{15}$$

One thing we can do with these eigenvectors is to normalize them, i.e., have it so that the sum of the squares of the entries is unity. Doing this yields

$$\bar{x}_1 = \left[\begin{array}{c} \frac{-1/2\,\sigma_y \tau_{xy}^{-1} + 1/2\,\sigma_x \tau_{xy}^{-1} + 1/2\,\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\tau_{xy}^{-1}}{\sqrt{1/2\,\frac{\sigma_y^2}{\tau_{xy}^2} - \frac{\sigma_x\sigma_y}{\tau_{xy}^2} - 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_y}{\tau_{xy}^2} + 1/2\,\frac{\sigma_x^2}{\tau_{xy}^2} + 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_x}{\tau_{xy}^2} + 2}} \\ \frac{1}{\sqrt{1/2\,\frac{\sigma_y^2}{\tau_{xy}^2} - \frac{\sigma_x\sigma_y}{\tau_{xy}^2} - 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_y}{\tau_{xy}^2} + 1/2\,\frac{\sigma_x^2}{\tau_{xy}^2} + 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_x}{\tau_{xy}^2} + 2}} \end{array}\right] \quad (16)$$

and

$$\bar{x}_2 = \left[\begin{array}{c} \frac{-1/2\,\sigma_y \tau_{xy}^{-1} + 1/2\,\sigma_x \tau_{xy}^{-1} - 1/2\,\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\tau_{xy}^{-1}}{\sqrt{1/2\,\frac{\sigma_y^2}{\tau_{xy}^2} - \frac{\sigma_x\sigma_y}{\tau_{xy}^2} + 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_y}{\tau_{xy}^2} + 1/2\,\frac{\sigma_x^2}{\tau_{xy}^2} - 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_x}{\tau_{xy}^2} + 2}} \\ \frac{1}{\sqrt{1/2\,\frac{\sigma_y^2}{\tau_{xy}^2} - \frac{\sigma_x\sigma_y}{\tau_{xy}^2} + 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_y}{\tau_{xy}^2} + 1/2\,\frac{\sigma_x^2}{\tau_{xy}^2} - 1/2\,\frac{\sqrt{\sigma_y^2 - 2\,\sigma_x\sigma_y + \sigma_x^2 + 4\,\tau_{xy}^2}\,\sigma_x}{\tau_{xy}^2} + 2}} \end{array}\right] \quad (17)$$

The eigenvectors, complicated as the algebra is, are useful in that we can diagonalize the matrix as follows:

$$S^{-1}AS = \Lambda \quad (18)$$

where

$$S = \left[\begin{array}{cc} \bar{x}_{1_1} & \bar{x}_{2_1} \\ \bar{x}_{1_2} & \bar{x}_{2_2} \end{array}\right] \quad (19)$$

This is referred to as a similarity or collinearity transformation. Similar matrices are matrices with the same eigenvalues. The matrix Λ, although simpler in form than A, has the same eigenvalues as the "original" matrix.

Either the original or the normalized forms of the eigenvectors can be used; the result will be the same as the scalar multiples will cancel in the matrix inversion. The normalization was done to illustrate the relationship between the eigenvectors, the diagonalization matrix, and the Givens rotation matrix, since for the last $sin^2\alpha + cos^2\alpha = 1$, an automatically normalized relationship. It is thus possible to extract the angle of the principal stresses from the eigenvectors.

Inverting the result of Equation 19 and multiplying through Equation 18 yields at last

$$\Lambda = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right] = \left[\begin{array}{cc} \sigma_1 & 0 \\ 0 & \sigma_3 \end{array}\right] \quad (20)$$

The two eigenvalues from Equations 12 and 13 are the principal stresses at the stress point in question. (The use of different subscripts for the eigenvalues and principal stresses comes from too many years dealing with Mohr-Coulomb failure theory.)

One practical result of Equations 9 and 20 and the underlying theory is that, for any coordinate orientation,

$$\sigma_x + \sigma_y = \sigma_1 + \sigma_3 \quad (21)$$

This is a very handy check when working problems such as this, especially since the algebra is so involved.

2.2. Two-Dimensional Example

To see all of this "in action" consider the soil stress states shown in Figure 2.

Consider the stress state "A." The stresses are expressed as a ratio of a pressure P at the surface; furthermore, following geotechnical engineering convention, compressive stresses are positive and the ordinate is the "z" axis.. For this study the convention of Figure _ will be adopted and thus $\sigma_x = -0.77$, $\sigma_y = -0.98$, $\tau_{xy} = 0.05$.

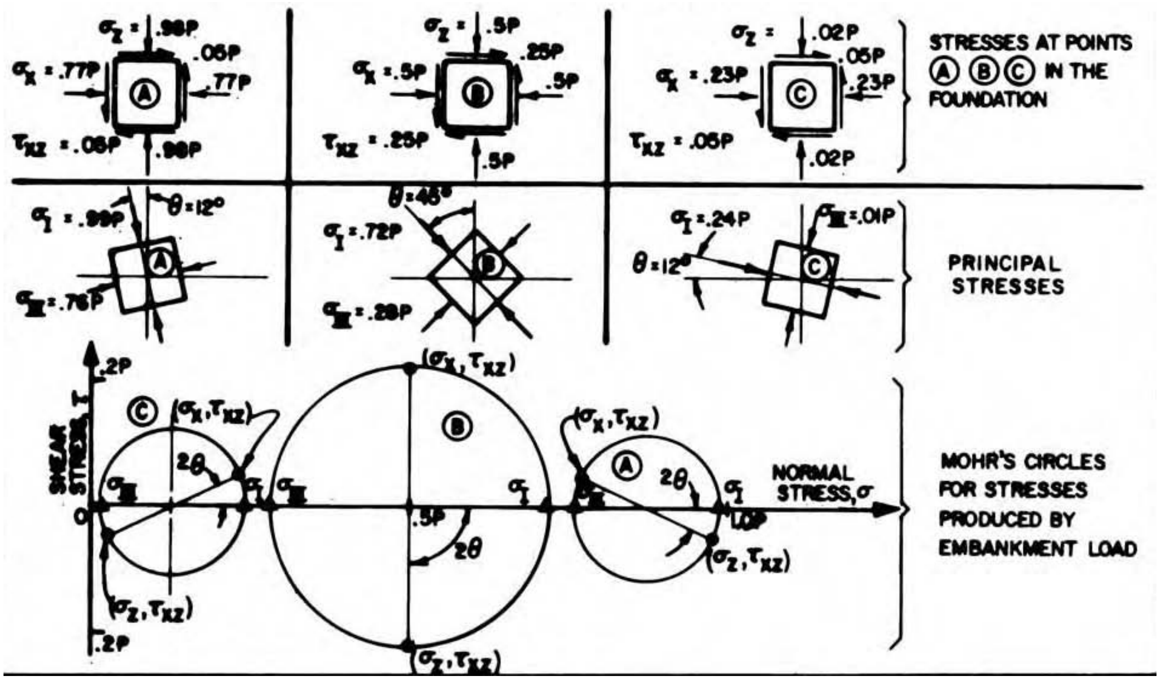

Figure 2: Stress State Example (from Navy [5])

By direct substitution (and carrying the results to precision unjustified in geotechnical engineering) we obtain the following:

- $\lambda_1 = \sigma_1 = -.7587029665P$ (Equations 12 and 20.)
- $\lambda_2 = \sigma_3 = -.9912970335P$ (Equations 12 and 20.) Note that the principal stresses are reversed; this is because the sign convention is reversed, and thus what was formerly the smaller of the stresses is now the larger. Also, the axis of the first principal stress changes because the first principal stress itself has changed.
- $S = \begin{bmatrix} .9754128670 & -.220385433 \\ .2203854365 & .9754128502 \end{bmatrix}$ (Equation 19.) Note that the normalized values of the eigenvectors (Equations 16 and 17) are used. Also note the similarity between this matrix and the Givens rotation matrix of Equation 5. The diagonalization process is simply a rotation process, as the physics of the problem suggest. (The diagonal terms should be equal to each other and the absolute values of the off-diagonal terms should be also; they are not because of numerical errors in Maple where they were computed.)
- $\alpha = 12.73167230°$; $\beta = 77.26832747°$ (Equations 2, 3 and 6.)
- In both cases the trace of A and Λ is the same, namely $1.75P$ (Equation 21.)

The results are thus the same. The precision is obviously greater than the "slide rule era" Navy [5], but the results illustrate that numerical errors can creep in, even with digital computation.

3. Three-Dimensional Analysis

3.1. Regular Principal Stresses

It is easy to see that, although the concepts involved are relatively simple, the algebra involved is involved. It is for this reason that the two-dimensional state was illustrated. With three dimensional stresses, the principles are the same, but the algebra is even more difficult.

To begin, one more direction cosine must be defined,

$$n = cos\gamma \tag{22}$$

where γ is of course the angle of the direction of the rotated coordinate system relative to the original one. The rotated system does not have to be the principal axis system, although that one is of most interest.

For three dimensions, Equation 4 should be written as follows:

$$\left[\begin{array}{ccc} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{array}\right]\left[\begin{array}{c} l \\ m \\ n \end{array}\right] = \left[\begin{array}{c} p_x \\ p_y \\ p_z \end{array}\right] \tag{23}$$

All of the moment summations of the shear stresses–which result in the symmetry of the matrix–have been included. The eigenvalues of the matrix are thus the solution of

$$\left[\begin{array}{ccc} \sigma_x - \lambda & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \lambda & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \lambda \end{array}\right] = 0 \tag{24}$$

The characteristic equation of this matrix–and thus the equation that solves for the eigenvalues–is

$$\lambda^3 - J_1\lambda^2 - J_2\lambda - J_3 = 0 \tag{25}$$

where

$$J_1 = \sigma_x + \sigma_y + \sigma_z \tag{26}$$

$$J_2 = \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 - (\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z) \tag{27}$$

$$J_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \left(\sigma_x\tau_{yz}^2 + \sigma_y\tau_{xz}^2 + \sigma_z\tau_{xy}^2\right) \tag{28}$$

It is easy to show the following:

- Equation 26 reduces to Equation 9 if $\sigma_z = 0$.
- Equation 27 reduces to Equation 10 if additionally $\tau_{xz} = \tau_{yz} = 0$.
- For all three conditions, Equation 25 reduces to 11.

From the previous derivation, we can thus expect

$$\lambda_1 = \sigma_1 \tag{29}$$

$$\lambda_2 = \sigma_2 \tag{30}$$

$$\lambda_3 = \sigma_3 \tag{31}$$

Solving a cubic equation such as this can be accomplished using the Cardan Formula. Generally, it is possible that we will end up with complex roots, but because of the properties of the matrix we are promised real roots. The eigenvalues are as follows:

$$\begin{aligned} \lambda_1 \;=\; & 1/6\sqrt[3]{36\,J_2J_1 - 108\,J_3 - 8\,{J_1}^3 + 12\sqrt{12\,{J_2}^3 - 3\,{J_2}^2{J_1}^2 - 54\,J_2J_1J_3 + 81\,{J_3}^2 + 12\,J_3{J_1}^3}} \\ & -6\,\frac{1/3\,J_2 - 1/9\,{J_1}^2}{\sqrt[3]{36\,J_2J_1 - 108\,J_3 - 8\,{J_1}^3 + 12\sqrt{12\,{J_2}^3 - 3\,{J_2}^2{J_1}^2 - 54\,J_2J_1J_3 + 81\,{J_3}^2 + 12\,J_3{J_1}^3}}} - 1/3\,J_1 \end{aligned} \tag{32}$$

$$\begin{aligned}\lambda_2 \quad = \quad & -1/12\,\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}\\ & +3\,\frac{1/3\,J_2-1/9\,{J_1}^2}{\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}}\\ & -1/3\,J_1+1/12\,\sqrt{-3}\,\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}\\ & +2\sqrt{-3}\,\frac{1/3\,J_2-1/9\,{J_1}^2}{\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}}\end{aligned} \tag{33}$$

$$\begin{aligned}\lambda_3 \quad = \quad & -1/12\,\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}\\ & +3\,\frac{1/3\,J_2-1/9\,{J_1}^2}{\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}}\\ & -1/3\,J_1-1/12\,\sqrt{-3}\,\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}\\ & -2\sqrt{-3}\,\frac{1/3\,J_2-1/9\,{J_1}^2}{\sqrt[3]{36\,J_2J_1-108\,J_3-8\,{J_1}^3+12\,\sqrt{12\,{J_2}^3-3\,{J_2}^2{J_1}^2-54\,J_2J_1J_3+81\,{J_3}^2+12\,J_3{J_1}^3}}}\end{aligned} \tag{34}$$

3.2. Deviatoric Principal Stresses

The algebra of Equations 32, 33 and 34 is rather involved. Is there a way to simplify it? The answer is "yes" if we use a common reduction of the Cardan Formula. Consider the following change of variables:

$$\sigma'_x = \sigma_x - \frac{J_1}{3} \tag{35}$$

$$\sigma'_y = \sigma_y - \frac{J_1}{3} \tag{36}$$

$$\sigma'_z = \sigma_z - \frac{J_1}{3} \tag{37}$$

Since the change is the same in all directions, we can write

$$\begin{bmatrix} \sigma'_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma'_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma'_z \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} \tag{38}$$

and

$$\begin{bmatrix} \sigma'_x - \lambda' & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma'_y - \lambda' & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma'_z - \lambda' \end{bmatrix} = 0 \tag{39}$$

The characteristic equation of the matrix in Equation 39 is

$$\lambda'^3 - J'_2\lambda' - J'_3 = 0 \tag{40}$$

where

$$J'_2 = \tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 - \left(\sigma'_x\sigma'_y + \sigma'_x\sigma'_z + \sigma'_y\sigma'_z\right) \tag{41}$$

$$J'_3 = \sigma'_x\sigma'_y\sigma'_z + 2\tau_{xy}\tau_{xz}\tau_{yz} - \left(\sigma'_x\tau_{yz}^2 + \sigma'_y\tau_{xz}^2 + \sigma'_z\tau_{xy}^2\right) \tag{42}$$

Although our motivation for this was to simplify the characteristic equation, the deviatoric stress in fact has physical significance. As Jaeger and Cook [4] point out, "...essentially $(\frac{I_1}{3})$determines uniform compression or dilation, while the stress deviation determines distortion. Since many criteria of failure are concerned primarily with distortion, and since they must be invariant with respect to rotation of axes, it appears that the invariants of stress deviation will be involved."

The one thing to be careful about is that the eigenvalues from Equation 24 are not the same as those from Equation 39. The latter are in fact "deviatoric eigenvalues" and are equivalent to deviatoric principal stresses, thus

$$\lambda_1' = \sigma_1' \tag{43}$$

$$\lambda_2' = \sigma_2' \tag{44}$$

$$\lambda_3' = \sigma_3' \tag{45}$$

These can be converted back to full stresses using Equations 35, 36 and 37. The eigenvalues for the deviatoric stresses are

$$\lambda_1' = 1/6\,\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}} + 2\,\frac{J_2'}{\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}}} \tag{46}$$

$$\begin{aligned}\lambda_2' \quad = \quad & -1/12\,\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}} - \frac{J_2'}{\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}}} \\ & +1/2\,\sqrt{-1}\sqrt{3}\left(1/6\,\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}} - 2\,\frac{J_2'}{\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}}}\right)\end{aligned} \tag{47}$$

$$\begin{aligned}\lambda_3' \quad = \quad & -1/12\,\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}} - \frac{J_2'}{\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}}} \\ & -1/2\,\sqrt{-1}\sqrt{3}\left(1/6\,\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}} - 2\,\frac{J_2'}{\sqrt[3]{108\,J_3' + 12\,\sqrt{-12\,{J_2'}^3 + 81\,{J_3'}^2}}}\right)\end{aligned} \tag{48}$$

Solving Equations 24 and 39 will yield eigenvalues and principal stresses of one kind or another. The eigenvectors and direction cosines can be determined using the same methods used for the two-dimensional problem. It should be readily evident, however, that the explicit solution of these eigenvectors and diagonalizing rotational matrices is very involved, although using deviatoric stresses simplifies the algebra considerably. If the material is isotropic, and its properties are the same in all directions, than the direction of the principal stresses can frequently be neglected. For anisotropic materials, this is not the case.

4. Finding the Eigenvalues and Eigenvectors Using the Method of Danilevsky

For the simple 3×3 matrix, computing the determinant, and from it the the invariants, is not a major task. With larger matrices, it is far more difficult to find the determinant, let alone the characteristic polynomial.

There have been many solutions to this problem over the years. For larger matrices the most common are Householder reflections and/or Givens rotations. We discussed the latter earlier. By doing multiple reflections and/or rotations, the matrix can be diagonalized and the eigenvalues “magically appear.” The whole problem of solving the characteristic equation is avoided, although the iterative cost in getting to the result can be considerable.

The method we’ll explore here is that of Danilevsky, first published in 1937, as presented in Faddeeva [3].

4.1. Theory

The math will be presented in 3×3 format, but it can be expanded to any size square matrix. The idea is, through a series of similarity transformations, to transform a 3×3 matrix (such as is shown in Equation 23) to the Frobenius normal form, which is

$$D = \begin{bmatrix} p_1 & p_2 & p_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{49}$$

From here we take the determinant of $D - \lambda I$, thus

$$D(\lambda) = \begin{vmatrix} p_1 - \lambda & p_2 & p_3 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} \tag{50}$$

The determinant–and thus the characteristic polynomial–is easy to compute, and setting it equal to zero,

$$\lambda^3 - \lambda^2 p_1 - p_2\lambda - p_3 = 0 \tag{51}$$

This is obviously the same as Equation 25, but now we can read the invariants straight out of the similar matrix.

So how do we get to Equation 49? The simple answer is through a series of row reductions. We start for the matrix A by carrying the third row into the second, dividing all the elements by the element in the last row by the element $a_{3,2}$, then subtracting the second column, multiplied by $a_{3,1}$, $a_{3,2}$, $a_{3,3}$ respectively from all the rest of the columns. Row and column manipulation is common with computational matrix routines, but a more straightforward way is to postmultiply A by the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{a_{3,1}}{a_{3,2}} & {a_{3,2}}^{-1} & -\frac{a_{3,3}}{a_{3,2}} \\ 0 & 0 & 1 \end{bmatrix} \tag{52}$$

The result will not be similar to A, but if we premultiply that result by M^{-1},

$$M^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} \\ 0 & 0 & 1 \end{bmatrix} \tag{53}$$

that result will be by Equation 19, although M does not diagonalize the result the way S did. Performing this series of matrix multplications yields

$$B = M^{-1}AM = \begin{bmatrix} a_{1,1} - \frac{a_{1,2}a_{3,1}}{a_{3,2}} & \frac{a_{1,2}}{a_{3,2}} & -\frac{a_{1,2}a_{3,3}}{a_{3,2}} + a_{1,3} \\ a_{3,1}\left(a_{1,1} - \frac{a_{1,2}a_{3,1}}{a_{3,2}}\right) + a_{3,2}\left(a_{2,1} - \frac{a_{2,2}a_{3,1}}{a_{3,2}}\right) & \frac{a_{1,2}a_{3,1}}{a_{3,2}} + a_{2,2} + a_{3,3} & a_{3,1}\left(-\frac{a_{1,2}a_{3,3}}{a_{3,2}} + a_{1,3}\right) + a_{3,2}\left(-\frac{a_{2,2}a_{3,3}}{a_{3,2}} + a_{2,3}\right) \\ 0 & 1 & 0 \end{bmatrix} \tag{54}$$

We continue this process by moving up a row, thus the new similarity transformation matrices are

$$N = \begin{bmatrix} {b_{2,1}}^{-1} & -\frac{b_{2,2}}{b_{2,1}} & -\frac{b_{2,3}}{b_{2,1}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{55}$$

$$N^{-1} = \begin{bmatrix} b_{2,1} & b_{2,2} & b_{2,3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{56}$$

Although we formed the M and N matrices first, from a conceptual and computational standpoint it is easier to form the "inverse" matrices first and then invert them. One drawback to Danilevsky's method is that there are many opportunities for division by zero, which need to be taken into consideration when employing the method.

In any case the next step yields

$$C = N^{-1}BN = \begin{bmatrix} b_{1,1} + b_{2,2} & b_{2,1}\left(-\frac{b_{1,1}b_{2,2}}{b_{2,1}} + b_{1,2}\right) + b_{2,3} & b_{2,1}\left(-\frac{b_{1,1}b_{2,3}}{b_{2,1}} + b_{1,3}\right) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{57}$$

From this Equation 49 is easily formed.

The coefficients $p(1)$, $p(2)$ and $p(3)$ are simply the invariants, which have already been established earlier for both standard and deviatoric normal stresses. From that standpoint the method seems to be overkill. The advantage comes when we compute the eigenvectors/direction cosines. Consider the vector

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \tag{58}$$

Premultiplying it by $D - \lambda I$ (see Equation 50) yields

$$\begin{bmatrix} p_1 - \lambda & p_2 & p_3 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} (p_1 - \lambda)\, y_{1,1} + p_2 y_{2,1} + p_3 y_{3,1} \\ y_{1,1} - \lambda\, y_{2,1} \\ y_{2,1} - \lambda\, y_{3,1} \end{bmatrix} = 0 \tag{59}$$

For y to be non-trivial,

$$y = \begin{bmatrix} \lambda^2 \\ \lambda \\ 1 \end{bmatrix} \tag{60}$$

It can be shown (Faddeeva [3]) that the eigenvectors x_n can be found as follows (successive rotations:)

$$\bar{x}_n = MNy = \begin{bmatrix} \frac{{\lambda_n}^2}{b_{2,1}} - \frac{b_{2,2}\lambda_n}{b_{2,1}} - \frac{b_{2,3}}{b_{2,1}} \\ -a_{3,1}\left(\frac{{\lambda_n}^2}{b_{2,1}} - \frac{b_{2,2}\lambda_n}{b_{2,1}} - \frac{b_{2,3}}{b_{2,1}}\right){a_{3,2}}^{-1} + \frac{\lambda_n}{a_{3,2}} - \frac{a_{3,3}}{a_{3,2}} \\ 1 \end{bmatrix} \tag{61}$$

These can, as was shown earlier, be normalized to direction cosines and can form the diagonalizing matrices.

4.2. Three-Dimensional Example

An example of this is drawn from Boresi et al. [1]. It involves a stress state, which referring to Equation 23 can be written (all stresses in kPa) as

$$A = \begin{bmatrix} 90 & -55 & -75 \\ -55 & 25 & 33 \\ -75 & 33 & -115 \end{bmatrix} \quad (62)$$

For the first transformation, we substitute these values into Equation 54 and thus

$$B = \begin{bmatrix} 1 & 0 & 0 \\ -75 & 33 & -85 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 120 & -55 & -75 \\ -55 & 55 & 33 \\ -75 & 33 & -85 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{25}{11} & 1/33 & \frac{85}{33} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -5/3 & -\frac{650}{3} \\ 2685 & 95 & 22014 \\ 0 & 1 & 0 \end{bmatrix} \quad (63)$$

The second transformation, following Equation 57,

$$C = \begin{bmatrix} 2685 & 95 & 22014 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & -5/3 & -\frac{650}{3} \\ 2685 & 95 & 22014 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2685} & -\frac{19}{537} & -\frac{7338}{895} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 90 & 18014 & -471680 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (64)$$

From this the characteristic equation is, from Equation 51,

$$\lambda^3 - 90\,\lambda^2 - 18014\,\lambda + 471680 = 0 \quad (65)$$

which yields eigenvalues/principal stresses of 176.8, 24.06 and -110.86 kPa.

So how does this look with deviatoric stresses? Applying Equations 26, 35, 36 and 37 yields

$$A' = \begin{bmatrix} 90 & -55 & -75 \\ -55 & 25 & 33 \\ -75 & 33 & -115 \end{bmatrix} \quad (66)$$

It's worth noting that, for the deviatoric stress matrix, the trace is always zero. The two similarity transformations are thus:

$$B' = \begin{bmatrix} 1 & 0 & 0 \\ -75 & 33 & -115 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 90 & -55 & -75 \\ -55 & 25 & 33 \\ -75 & 33 & -115 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{25}{11} & 1/33 & \frac{115}{33} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -35 & -5/3 & -\frac{800}{3} \\ 2685 & 35 & 23964 \\ 0 & 1 & 0 \end{bmatrix} \quad (67)$$

$$C' = \begin{bmatrix} 2685 & 35 & 23964 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -35 & -5/3 & -\frac{800}{3} \\ 2685 & 35 & 23964 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2685} & -\frac{7}{537} & -\frac{7988}{895} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 20714 & 122740 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (68)$$

and the characteristic polynomial (missing the squared term, as expected)

$$\lambda^3 - 20714\,\lambda - 122740 = 0 \quad (69)$$

which yields eigenvalues/principal deviatoric stresses of 146.8, -5.94 and -140.86 kPa.

5. Numerical Implementation

Even with a "simple" problem such as this, the algebra of the solution can become very involved. This is a problem where mistakes are easy to make with hand solutions (something that perusal of older textbooks will make abundantly clear.) For problems such as finite element analysis, accurate and robust solution on a large scale of these equations is necessary.

The original, FORTRAN 77 routine to solve this problem performed the steps as outlined above but did not address two crucial lacunae in Danilevsky's Method:

1. Solving the characteristic polynomial in a way that gives consistent results.

2. Dealing with matrices where one or more of the elements adjacent to the diagonal (and in some cases others) are zero. By arriving at an explicit solution to the invariants (i.e., multiplying through all the steps of the Danilevsky Method) we eliminated the division by zero problem for this stage of the solution, but not for the determination of the eigenvectors.

We should first address these issues.

5.1. Solution of the Characteristic Polynomial

The original code used Newton's Method to solve for these, using the diagonal elements as starting values. In some cases this worked but in others one would start with different seed values and end up with the same eigenvalue. The Cardan Formula solution is shown above, but a simpler way to implement this is through a trigonometric transformation, which is shown in Figure 3.

The principal deviatoric stresses σ_1', σ_2', σ_3' are given as the roots of the cubic equation[11]

$$t^3 - J_2' t - J_3' = 0. \qquad (7.57)$$

Noting the trigonometric identity

$$\sin^3\theta - \tfrac{3}{4}\sin\theta + \tfrac{1}{4}\sin 3\theta = 0, \qquad (7.58)$$

and substituting $t = r\sin\theta$ into (7.57) we have

$$\sin^3\theta - \frac{J_2'}{r^2}\sin\theta - \frac{J_3'}{r^3} = 0. \qquad (7.59)$$

Comparing (7.58) and (7.59) gives

$$r = \frac{2}{\sqrt{3}}(J_2')^{1/2}, \qquad (7.60)$$

$$\sin 3\theta = -\frac{4J_3'}{r^3} = -\frac{3\sqrt{3}}{2}\,\frac{J_3'}{(J_2')^{3/2}}. \qquad (7.61)$$

The first root of (7.61) with θ determined for 3θ in the range $\pm\pi/2$ is a convenient alternative to the third invariant, J_3'. By noting the cyclic nature of $\sin(3\theta + 2n\pi)$ we have immediately the three (and only three) possible values of $\sin\theta$ which define the three principal stresses. The deviatoric principal stresses are given by $t = r\sin\theta$ on substitution of the three values of $\sin\theta$ in turn. Substituting for r from (7.60) and adding the mean hydrostatic stress component gives the total principal stresses to be

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{Bmatrix} = \frac{2(J_2')^{\frac{1}{2}}}{\sqrt{3}}\begin{Bmatrix}\sin\left(\theta + \frac{2\pi}{3}\right)\\ \sin\theta\\ \sin\left(\theta + \frac{4\pi}{3}\right)\end{Bmatrix} + \frac{J_1}{3}\begin{Bmatrix}1\\ 1\\ 1\end{Bmatrix}, \qquad (7.62)$$

with $\sigma_1 > \sigma_2 > \sigma_3$ and $-\pi/6 \leqslant \theta \leqslant \pi/6$. The term θ is essentially similar to the Lode parameter[1] Γ defined by $\Gamma = -\sqrt{(3)}\tan\theta$.

Figure 3: Trigonometric Transformation and Principal Stresses (from Owen and Hinton [6])

This result also gives us the Lode Angle θ, which is important in failure theory.

One important exception to this method was implemented for cases when $p_3 = 0$ either for the regular or deviatoric cases. For these cases Equation 51 reduces to the quadratic case where one of the eigenvalues/principal stresses is zero and the other two can be solved using Equations 12 and 13. This is a more general test than, say, testing the normal stresses.

5.2. Solving for the Eigenvectors

For "full" matrices (i.e., matrices with all non-zero elements) Danilevsky's Method is computationally efficient and accurate. For those where any of the elements adjacent to the diagonal are zero, Faddeeva

[3] suggests using a row and column interchange technique. A general test for this problem is if either of Equations 70 is true:

$$\begin{aligned} \tau_{yz} &= 0 \\ \tau_{xy}\left(\tau_{xz}^2 - \tau_{yz}^2\right) + \tau_{xz}\tau_{yz}\left(\sigma_y - \sigma_x\right) &= 0 \end{aligned} \tag{70}$$

A more general alternative to this–for all cases, really–is given in DeRusso et al. [2].

They begin by considering the adjoint of Equation 24, except that the diagonal entries are subtracted from the eigenvalues and not the other way around as is the case in Equation 24. The result of of this is

$$A_{Adj} = \begin{bmatrix} \lambda^2 - \lambda\,\sigma_z - \sigma_y\,\lambda + \sigma_y\,\sigma_z - {\tau_{yz}}^2 & \tau_{xy}\,\lambda - \tau_{xy}\,\sigma_z + \tau_{yz}\,\tau_{xz} & \tau_{xy}\,\tau_{yz} + \tau_{xz}\,\lambda - \tau_{xz}\,\sigma_y \\ \tau_{xy}\,\lambda - \tau_{xy}\,\sigma_z + \tau_{yz}\,\tau_{xz} & \lambda^2 - \lambda\,\sigma_z - \sigma_x\,\lambda + \sigma_x\,\sigma_z - {\tau_{xz}}^2 & \tau_{yz}\,\lambda - \tau_{yz}\,\sigma_x + \tau_{xy}\,\tau_{xz} \\ \tau_{xy}\,\tau_{yz} + \tau_{xz}\,\lambda - \tau_{xz}\,\sigma_y & \tau_{yz}\,\lambda - \tau_{yz}\,\sigma_x + \tau_{xy}\,\tau_{xz} & \lambda^2 - \sigma_y\,\lambda - \sigma_x\,\lambda + \sigma_x\,\sigma_y - {\tau_{xy}}^2 \end{bmatrix} \tag{71}$$

If the values of λ (or the principal stresses) are successively substituted into Equation 71 (the other parameters are already known) we end up with three matrices. Each matrix has three columns, each of which are the eigenvectors for a particular principal stress. To obtain the direction cosines we must normalize these values (as was the case with Danilevsky's Method) by dividing each entry in the eigenvector by the sum of the squares of all of the entries in the eigenvector. This takes advantage of the fact that an eigenvector can be multiplied by any scalar and remain a valid eigenvalue.

The "trout in the milk" is that "any scalar" includes 0 and -1, which can create problems for us. Since 0 is the more serious problem, we will consider it first.

In obtaining the result of Equation 71, it is entirely possible (and a valid result) to obtain a column of zeros for the eigenvector. This may be a valid result, but it isn't very useful; it is necessary to skip over such columns and find one which can yield a non-zero result.

Turning to -1, one common result is that the eigenvectors/direction cosines can have opposite signs from those obtained with hand, non-linear algebra solutions. This is a product of the nature of the eigenvalues. We saw this in the example given above and it must be kept in mind when reviewing the results. This is true for both methods; in fact, the adjoint method can yield two eigenvectors in one matrix which are simply the negatives of each other.

5.3. Actual Code Implementation

As mentioned earlier, the original routine was coded in FORTRAN 77. The revised routine was coded in php and is hosted on a web server.[2] It is capable of solving the problem both for standard and deviatoric stresses, the user selects which one at input. The steps used in broad outline are as follows:

1. If deviatoric stresses are specified, the normal stresses are accordingly converted as shown earlier.
2. The invariants are computed. For a more general routine of Danilevsky's Method, the matrices associated with the similarity transformations–and their inverses–would have to be computed and the matrix multiplications done. In this case the results of these multiplications–the invariants themselves–are "hard coded" into the routine, which saves a great deal of matrix multiplication. It also eliminated problems with division by zero up to this point.
3. The eigenvalues/principal stresses are computed using the method shown in Figure 3, unless $p_3 = 0$, in which case the non-zero eigenvalues/principal stresses are computed quadratically. (For characteristic polynomials beyond degree four, another type of solution is necessary.)
4. Taking the three resulting eigenvalues, the eigenvectors were computed either using Equation 61 or the results of Equation 71. For the first method the matrix multiplications were hard-coded into the routine. The eigenvectors were then normalized, which yielded three sets of direction cosines.

[2] Currently it is hosted at http://www.paludavia.com/tamwave/danilevsky/.

We will present two examples, both of which are from Boresi et al. [1].

5.3.1. Example 1

This is the example shown earlier. The input page is shown in Figure 4.

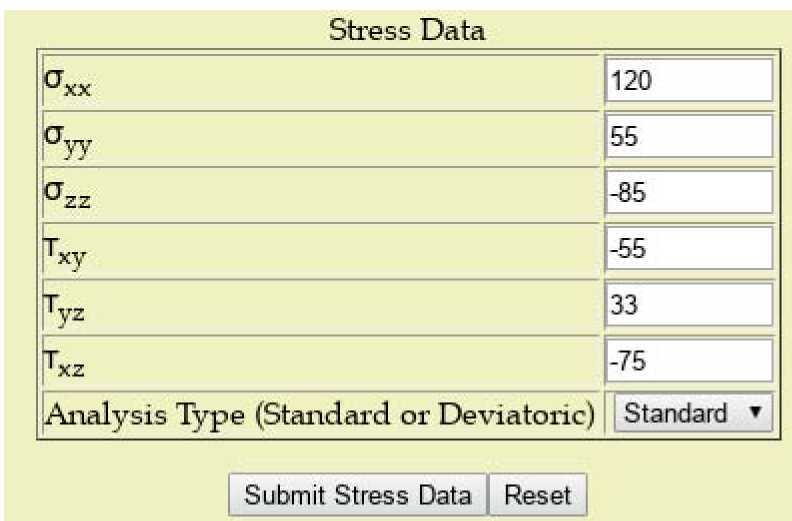

Figure 4: Input for Example 1

The results for the regular stresses are shown in Figure 5 and those for deviatoric stresses are shown in Figure 6.

Stress State Using Danilevsky's Method
Regular Normal Stresses

Original Stress Matrix/Tensor		
120.0	-55.0	-75.0
-55.0	55.0	33.0
-75.0	33.0	-85.0
Invariants		
90.0	18,014.0	-471,680.0
Principal Stresses		
176.8	24.1	-110.9
Maximum Shear Stress: 143.8		
Lode Angle, Degrees: -2.0		
Direction Cosines		
-0.837	-0.465	0.287
0.459	-0.884	-0.094
0.298	0.053	0.953
Angles from Direction Cosines, Degrees		
146.8	117.7	73.3
62.7	152.1	95.4
72.7	87.0	17.6

Figure 5: Regular Normal Stress Results for Test Case 1

Stress State Using Danilevsky's Method
Deviatoric Normal Stresses

Original Stress Matrix/Tensor		
90.0	-55.0	-75.0
-55.0	25.0	33.0
-75.0	33.0	-115.0
Invariants		
0.0	20,714.0	122,740.0
Principal Stresses		
146.8	-5.9	-140.9
Maximum Shear Stress: 143.8		
Lode Angle, Degrees: -2.0		
Direction Cosines		
-0.837	-0.465	0.287
0.459	-0.884	-0.094
0.298	0.053	0.953
Angles from Direction Cosines, Degrees		
146.8	117.7	73.3
62.7	152.1	95.4
72.7	87.0	17.6

Figure 6: Deviatoric Normal Stress Results for Test Case 1

The differences between the regular and deviatoric stress cases are as follows:

1. The diagonal entries/normal stresses.
2. Invariants (the first invariant of the deviatoric case is, of course, zero.)
3. Principal stresses.

The principal stress results agree with the "hand" calculations earlier. The Lode Angle, maximum shear stress, and the direction cosines–and thus the diagonalizing matrices–are the same for both cases. Note that the angles from the direction cosines are "naïve" in that they simply use the arc-cosine function of php and do not take into account any periodic effects.

5.3.2. Example 2

The input parameters for this case are shown in Figure 7.

The results for the regular stresses are shown in Figure 8 and those for deviatoric stresses are shown in Figure 9.

Stress Data

σ_{xx}	110
σ_{yy}	-86
σ_{zz}	55
τ_{xy}	60
τ_{yz}	0
τ_{xz}	0
Analysis Type (Standard or Deviatoric)	Deviatoric ▼

Submit Stress Data | Reset

Figure 7: Input for Example 2

Stress State Using Danilevsky's Method
Regular Normal Stresses

Original Stress Matrix/Tensor		
110.0	60.0	0.0
60.0	-86.0	0.0
0.0	0.0	55.0
Invariants		
79.0	11,740.0	-718,300.0
Principal Stresses		
126.9	55.0	-102.9
Maximum Shear Stress: 114.9		
Lode Angle, Degrees: 12.2		
Direction Cosines (Adjoint Method)		
0.963	-0.920	0.271
0.271	-0.392	-0.963
0.000	0.000	0.000
Angles from Direction Cosines, Degrees		
15.7	156.9	74.3
74.3	113.1	164.3
90.0	90.0	90.0

Figure 8: Regular Normal Stress Results for Test Case 2

Stress State Using Danilevsky's Method Deviatoric Normal Stresses		
Original Stress Matrix/Tensor		
83.7	60.0	0.0
60.0	-112.3	0.0
0.0	0.0	28.7
Invariants		
0.0	13,820.3	-372,625.3
Principal Stresses		
100.6	28.7	-129.2
Maximum Shear Stress: 114.9		
Lode Angle, Degrees: 12.2		
Direction Cosines (Adjoint Method)		
0.963	-0.920	0.271
0.271	-0.392	-0.963
0.000	0.000	0.000
Angles from Direction Cosines, Degrees		
15.7	156.9	74.3
74.3	113.1	164.3
90.0	90.0	90.0

Figure 9: Deviatoric Normal Stress Results for Test Case 2

Note that the adjoint method is being used and is noted.

6. Conclusion

We have explored the analysis of Mohr's Circle for stress states in both two and three dimensions. The equations can be derived from strictly linear algebra considerations, the principal stresses being the eigenvalues and the direction cosines being the normalized eigenvectors. It was also shown that numerical methods can be employed to analyze these results and the invariants, in this case using the computationally efficient Danilevsky's Method except in certain cases where an adjoint method was used for the eigenvectors.

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