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SIMPLIFIED COMPUTATION OF VERTICAL PRESSURES IN ELASTIC FOUNDATIONS

BY

NATHAN M. NEWMARK

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SIMPLIFIED COMPUTATION OF VERTICAL PRESSURES IN ELASTIC FOUNDATIONS

I. Introduction

1. *Introductory.*—In estimating the probable settlement under a loaded area, as under a building, it is necessary to determine the pressure distribution at various points in the foundation. It is suggested in the "Progress Report of the Special Committee on Earths and Foundations," Proc. A.S.C.E., May, 1933, pp. 777-820, that the distribution of vertical stresses on horizontal planes in soil is given to a sufficiently close approximation for practical purposes by Boussinesq's formula for the stress distribution in a homogeneous, elastic, isotropic body of semi-infinite extent bounded by a plane and loaded by forces perpendicular to that plane. Settlements predicted on the basis of pressures so computed are said to have agreed fairly well with observed settlements.*

The general procedure for computing pressure due to a given load is to divide the loaded area into elements sufficiently small to permit the assumption that the load on the element of area is concentrated at a point. Then by use of Boussinesq's formula for the vertical stress due to a concentrated load, the total stresses are computed as the sum of the individual stresses due to the separate concentrations. This process becomes rather tedious and involves considerable time, but cannot be avoided when the loads are irregular.

However, it is possible to integrate Boussinesq's formula to obtain the stress distribution due to a load uniformly distributed over a rectangle. The result is fairly simple, but more important, can readily be tabulated in such a way that the stress due to loads distributed over any combination of rectangular areas can quickly and easily be determined.

Integrations of Boussinesq's formula to obtain vertical stress distributions for a load distributed along a line of infinite extent, for a load uniformly distributed on a strip of constant width and infinite length, and for a load uniformly distributed over a circular area, as well as other more complicated cases of loading, have appeared in the literature,† but so far as the writer can determine, the solution

*Proc. A.S.C.E., May, 1933, p. 798, p. 808.

†For example, see Timoshenko, "Theory of Elasticity," McGraw-Hill, 1934, p. 82, 333; Terzaghi, "Erdbaumechanik," Franz Deuticke, Vienna, 1925, p. 226; A. and L. Föppl, "Drang und Zwang," R. Oldenbourg, Berlin, 2nd edition, 1924, Vol. II, p. 230; and for a resumé of a number of cases, "The Application of Theories of Elasticity and Plasticity to Foundation Problems," Leo Jürgenson, Journal of the Boston Society of Civil Engineers, July, 1934, p. 229.

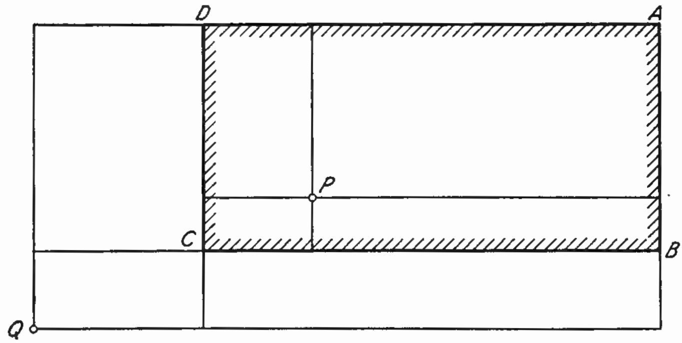

FIG. 1. DIVISION OF AREA *ABCD* INTO COMPONENTS FOR COMPUTATION OF STRESS AT *P* AND *Q*

for the case of a load distributed over a rectangle has not previously been published.

By use of the formulas derived, a table has been computed giving the pressure in terms of the intensity of load at a point a unit depth below the corner of a rectangular area uniformly loaded. Various examples illustrating the use of the table are given herein.

In view of the fact that computations of this nature can at best be only approximate it seems reasonable, in general, that the loads to be treated may be taken as uniformly distributed over rectangular areas. Then by combining various cases the pressure at any point in the foundation, for practically any loading, can be obtained from the values given in the table.

2. *Acknowledgment.*—The investigation herein described was conducted as part of the work of the Engineering Experiment Station of the University of Illinois, under the general administrative direction of DEAN M. L. ENGER, Director of the Engineering Experiment Station, and of PROF. W. C. HUNTINGTON, Head of the Department of Civil Engineering.

II. VERTICAL STRESS DUE TO LOAD UNIFORMLY DISTRIBUTED OVER A RECTANGLE

3. *Stress at Any Depth Under Corner of Uniformly Loaded Rectangle.*—Suppose that in Fig. 1 it is desired to find the intensity of vertical stress (or pressure) at a point at a given distance vertically below the point P in the loaded rectangle. The intensity of stress at P may be considered as equal to the sum of the stress intensities at

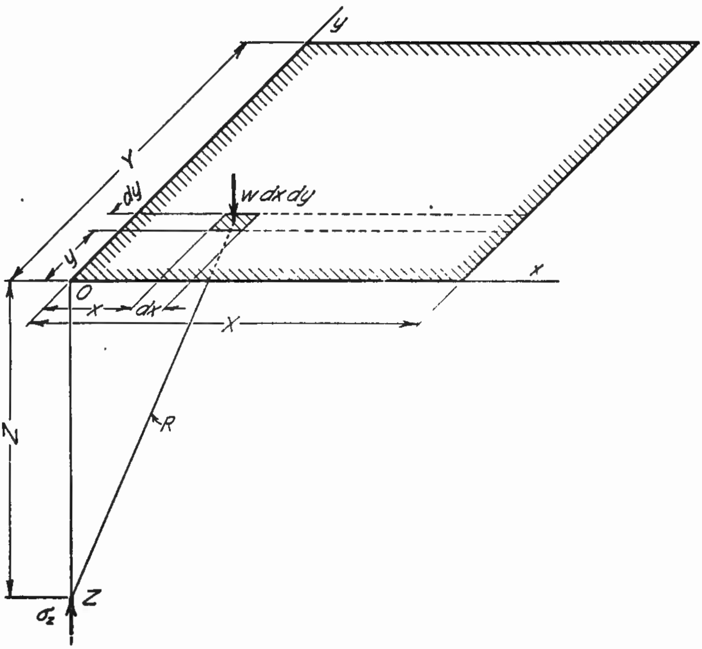

FIG. 2. DIAGRAM SHOWING NOTATION USED IN FORMULAS

P due to the loads on the four rectangles $PA + PB + PC + PD$. For the point Q outside of the load the intensity of vertical stress at a given depth may be considered as due to loaded rectangles as follows: $QA - QB - QD + QC$. Thus, the pressure at any point due to a load uniformly distributed over a rectangle may be found by combining not more than four cases of the stress under the corner of a loaded rectangle.

A formula for this stress may be obtained as follows: In Fig. 2 let the xy plane be the surface of the foundation, and let the load be applied over the rectangle XOY with an intensity of w per unit area. It is desired to find the vertical stress, σ_z, at a depth Z on the z axis. The element of area $dxdy$ carries a load $dp = wdxdy$. The increment of pressure at Z due to this element of load is $d\sigma_z$. Denoting by R the radius vector from Z to the element of area, using Boussinesq's formula,*

$$\sigma_z = \frac{3P}{2\pi}\frac{z^3}{R^5} \tag{1}$$

*See, for example, Timoshenko, "Theory of Elasticity," p. 331.

for the intensity of vertical stress due to a load P, and considering a load $dP = w dx dy$ on the element of area, the intensity of vertical stress at the point under consideration is

$$d\sigma_z = \frac{3w}{2\pi} \frac{Z^3}{R^5} dx dy \tag{2}$$

Taking the summation of this expression over the entire area, and putting $R^2 = x^2 + y^2 + Z^2$,

$$\sigma_z = \frac{3w}{2\pi} \int_0^Y \int_0^X \frac{Z^3 \, dx}{(x^2 + y^2 + Z^2)^{5/2}} dy \tag{3}$$

The integration and simplification required for the evaluation of this expression, although not difficult, is quite tedious, and will not be given here.* The resulting formula for the stress is

$$\sigma_z = \frac{w}{4\pi} \left[\frac{2XYZ (X^2 + Y^2 + Z^2)^{1/2}}{Z^2 (X^2 + Y^2 + Z^2) + X^2Y^2} \cdot \frac{X^2 + Y^2 + 2Z^2}{X^2 + Y^2 + Z^2} + \left\{ \begin{matrix} \tan^{-1} \dfrac{2XYZ (X^2 + Y^2 + Z^2)^{1/2}}{Z^2 (X^2 + Y^2 + Z^2) - X^2Y^2} \\ \text{or} \\ \sin^{-1} \dfrac{2XYZ (X^2 + Y^2 + Z^2)^{1/2}}{Z^2 (X^2 + Y^2 + Z^2) + X^2Y^2} \end{matrix} \right\} \right] \tag{4}$$

The first term within the brackets is always positive, and has a value ranging between zero and about 1.2. Both forms for the second term have the same value. The value of the second term is always positive, and ranges between 0 and π. When $Z^2 (X^2 + Y^2 + Z^2)$ is greater than X^2Y^2, the second term is less than $\frac{\pi}{2}$, and when it is less than X^2Y^2, the second term is between $\frac{\pi}{2}$ and π. It will be

*The following integrals obtained in the process of evaluating Equation (3) may be verified by differentiation:

$$\int \frac{dx}{(x^2 + y^2 + Z^2)^{5/2}} = \frac{x (2x^2 + 3y^2 + 3Z^2)}{3 (y^2 + Z^2)^2 (x^2 + y^2 + Z^2)^{3/2}}.$$

$$\int \frac{2Z^3X (2X^2 + 3y^2 + 3Z^2) \, dy}{(y^2 + Z^2)^2 (X^2 + y^2 + Z^2)^{3/2}} =$$

$$\frac{2XyZ (X^2 + y^2 + 2Z^2)}{(X^2 + Z^2)(y^2 + Z^2)(X^2 + y^2 + Z^2)^{1/2}} + \tan^{-1} \frac{2XyZ (X^2 + y^2 + Z^2)^{1/2}}{(X^2 + Z^2)(y^2 + Z^2) - 2X^2y^2}$$

noted that X and Y may be interchanged without affecting the value of σ_z.

The formula may be written in a more convenient form by putting V = ratio of the radius vector from point Z to the far corner of the loaded area, to the depth Z, or

$$V^2 = \frac{X^2 + Y^2 + Z^2}{Z^2},$$

and putting A = ratio of the area of the loaded rectangle to the square of the depth Z, or $A = \frac{XY}{Z^2}$; then,

$$\sigma_z = \frac{w}{4\pi}\left[\frac{2AV}{V^2 + A^2}\cdot\frac{V^2+1}{V^2} + \left\{\begin{matrix} \tan^{-1}\dfrac{2AV}{V^2 - A^2} \\ \text{or} \\ \sin^{-1}\dfrac{2AV}{V^2 + A^2}\end{matrix}\right\}\right] \qquad (5)$$

If $\frac{X}{Z} = m$, and $\frac{Y}{Z} = n$ be taken as the relative dimensions of the loaded area, then $A = mn$, and $V^2 = m^2 + n^2 + 1$. The quantity in the brackets is a pure number, and σ_z and w are expressed in the same units.

4. *Limiting Cases.*—It is of interest to examine the formulas for several limiting cases. Consider the stress at depth Z under the center of a rectangle of sides $2X$ by $2Y$. This will be four times the values given in Equations (4) and (5).

Case I. When Z approaches zero, from Equation (4), the stress becomes

$$\sigma_z = \frac{w}{\pi}\left[0 + \tan^{-1}(-Z\cdot C)\right] = \frac{w}{\pi}(\pi) = w.$$

That is, the stress directly under the load is equal to the load.

From Equation (5), if X and Y approach infinite values, the effect is the same as if Z approaches zero. Hence $\sigma_z = w$ at all finite depths under a load distributed over an infinite area, as would be expected.

Case II. When m and n are very small, that is, when Z becomes

TABLE 1
VERTICAL PRESSURE AT UNIT DEPTH UNDER CORNER OF RECTANGLE OF DIMENSIONS *m* by *n*, LOADED UNIFORMLY

Values are for $\frac{\sigma}{w}$

m	*n*											
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.2	1.4
0.1	0.00470	0.00917	0.01323	0.01678	0.01978	0.02223	0.02420	0.02576	0.02698	0.02794	0.02926	0.03007
0.2	0.00917	0.01790	0.02585	0.03280	0.03866	0.04348	0.04735	0.05042	0.05283	0.05471	0.05733	0.05894
0.3	0.01323	0.02585	0.03735	0.04742	0.05593	0.06294	0.06858	0.07308	0.07661	0.07938	0.08323	0.08561
0.4	0.01678	0.03280	0.04742	0.06024	0.07111	0.08009	0.08734	0.09314	0.09770	0.10129	0.10631	0.10941
0.5	0.01978	0.03866	0.05593	0.07111	0.08403	0.09473	0.10340	0.11035	0.11584	0.12018	0.12626	0.13003
0.6	0.02223	0.04348	0.06294	0.08009	0.09473	0.10688	0.11679	0.12474	0.13105	0.13605	0.14309	0.14749
0.7	0.02420	0.04735	0.06858	0.08734	0.10340	0.11679	0.12772	0.13653	0.14356	0.14914	0.15703	0.16199
0.8	0.02576	0.05042	0.07308	0.09314	0.11035	0.12474	0.13653	0.14607	0.15371	0.15978	0.16843	0.17389
0.9	0.02698	0.05283	0.07661	0.09770	0.11584	0.13105	0.14356	0.15371	0.16185	0.16835	0.17766	0.18357
1.0	0.02794	0.05471	0.07938	0.10129	0.12018	0.13605	0.14914	0.15978	0.16835	0.17522	0.18508	0.19139
1.2	0.02926	0.05733	0.08323	0.10631	0.12626	0.14309	0.15703	0.16843	0.17766	0.18508	0.19584	0.20278
1.4	0.03007	0.05894	0.08561	0.10941	0.13003	0.14749	0.16199	0.17389	0.18357	0.19139	0.20278	0.21020
1.6	0.03058	0.05994	0.08709	0.11135	0.13241	0.15028	0.16515	0.17739	0.18737	0.19546	0.20731	0.21510
1.8	0.03090	0.06058	0.08804	0.11260	0.13395	0.15207	0.16720	0.17967	0.18986	0.19814	0.21032	0.21836
2.0	0.03111	0.06100	0.08867	0.11342	0.13496	0.15326	0.16856	0.18119	0.19152	0.19994	0.21235	0.22058
2.5	0.03138	0.06155	0.08948	0.11450	0.13628	0.15483	0.17036	0.18321	0.19375	0.20236	0.21512	0.22364
3.0	0.03150	0.06178	0.08982	0.11495	0.13684	0.15550	0.17113	0.18407	0.19470	0.20341	0.21633	0.22499
4.0	0.03158	0.06194	0.09007	0.11527	0.13724	0.15598	0.17168	0.18469	0.19540	0.20417	0.21722	0.22600
5.0	0.03160	0.06199	0.09014	0.11537	0.13737	0.15612	0.17185	0.18488	0.19561	0.20440	0.21749	0.22632
6.0	0.03161	0.06201	0.09017	0.11541	0.13741	0.15617	0.17191	0.18496	0.19569	0.20449	0.21760	0.22644
8.0	0.03162	0.06202	0.09018	0.11543	0.13744	0.15621	0.17195	0.18500	0.19574	0.20455	0.21767	0.22652
10.0	0.03162	0.06202	0.09019	0.11544	0.13745	0.15622	0.17196	0.18502	0.19576	0.20457	0.21769	0.22654
∞	0.03162	0.06202	0.09019	0.11544	0.13745	0.15623	0.17197	0.18502	0.19577	0.20458	0.21770	0.22656

TABLE 1 (Concluded)

m	n											
	1.4	1.6	1.8	2.0	2.5	3.0	4.0	5.0	6.0	8.0	10.0	∞
0.1	0.03007	0.03058	0.03090	0.03111	0.03138	0.03150	0.03158	0.03160	0.03161	0.03162	0.03162	0.03162
0.2	0.05894	0.05994	0.06058	0.06100	0.06155	0.06178	0.06194	0.06199	0.06201	0.06202	0.06202	0.06202
0.3	0.08561	0.08709	0.08804	0.08867	0.08948	0.08982	0.09007	0.09014	0.09017	0.09018	0.09019	0.09019
0.4	0.10941	0.11135	0.11260	0.11342	0.11450	0.11495	0.11527	0.11537	0.11541	0.11543	0.11544	0.11544
0.5	0.13003	0.13241	0.13395	0.13496	0.13628	0.13684	0.13724	0.13737	0.13741	0.13744	0.13745	0.13745
0.6	0.14749	0.15028	0.15207	0.15326	0.15483	0.15550	0.15598	0.15612	0.15617	0.15621	0.15622	0.15623
0.7	0.16199	0.16515	0.16720	0.16856	0.17036	0.17113	0.17168	0.17185	0.17191	0.17195	0.17196	0.17197
0.8	0.17389	0.17739	0.17967	0.18119	0.18321	0.18407	0.18469	0.18488	0.18496	0.18500	0.18502	0.18502
0.9	0.18357	0.18737	0.18986	0.19152	0.19375	0.19470	0.19540	0.19561	0.19569	0.19574	0.19576	0.19577
1.0	0.19139	0.19546	0.19814	0.19994	0.20236	0.20341	0.20417	0.20440	0.20449	0.20455	0.20457	0.20458
1.2	0.20278	0.20731	0.21032	0.21235	0.21512	0.21633	0.21722	0.21749	0.21760	0.21767	0.21769	0.21770
1.4	0.21020	0.21510	0.21836	0.22058	0.22364	0.22499	0.22600	0.22632	0.22644	0.22652	0.22654	0.22656
1.6	0.21510	0.22025	0.22372	0.22610	0.22940	0.23088	0.23200	0.23236	0.23249	0.23258	0.23261	0.23263
1.8	0.21836	0.22372	0.22736	0.22986	0.23334	0.23495	0.23698	0.23735	0.23671	0.23681	0.23684	0.23686
2.0	0.22058	0.22610	0.22986	0.23247	0.23614	0.23782	0.23912	0.23954	0.23970	0.23981	0.23985	0.23987
2.5	0.22364	0.22940	0.23334	0.23614	0.24010	0.24196	0.24344	0.24392	0.24412	0.24425	0.24429	0.24432
3.0	0.22499	0.23088	0.23495	0.23782	0.24196	0.24394	0.24554	0.24608	0.24630	0.24646	0.24650	0.24654
4.0	0.22600	0.23200	0.23698	0.23912	0.24344	0.24554	0.24729	0.24791	0.24817	0.24836	0.24842	0.24846
5.0	0.22632	0.23236	0.23735	0.23954	0.24392	0.24608	0.24791	0.24857	0.24885	0.24907	0.24914	0.24919
6.0	0.22644	0.23249	0.23671	0.23970	0.24412	0.24630	0.24817	0.24885	0.24916	0.24939	0.24946	0.24952
8.0	0.22652	0.23258	0.23681	0.23981	0.24425	0.24646	0.24836	0.24907	0.24939	0.24964	0.24973	0.24980
10.0	0.22654	0.23261	0.23684	0.23985	0.24429	0.24650	0.24842	0.24914	0.24946	0.24973	0.24981	0.24989
∞	0.22656	0.23263	0.23686	0.23987	0.24432	0.24654	0.24846	0.24919	0.24952	0.24980	0.24989	0.25000

large or X and Y are very small, then A is small, and V approaches unity. From Equation (5)

$$\sigma_z = \frac{w}{\pi}\left[2A \cdot 2 + \tan^{-1} 2A\right]$$

and $\tan^{-1} 2A$ approaches $2A$ when A is small, and $\sigma_z = \frac{w}{\pi}(6A)$. But the total load P is $2X \cdot 2Y \cdot w = 4AZ^2w$, or $A = \frac{P}{4wZ^2}$. Substituting,

$$\sigma_z = \frac{3}{2}\frac{P}{\pi}\cdot\frac{1}{Z^2} \tag{6}$$

which checks the result from Boussinesq's formula, Equation (1), when $R = Z$, or for points under the line of action of the load.

Case III. When m approaches infinity we have the case of a strip of infinite length and of width $2Y$. For this case $\frac{A}{m} = n$, $\frac{V}{m} = 1$, $\frac{1}{m} = 0$, and the stress becomes

$$\sigma_z = \frac{w}{\pi}\left[\frac{2n}{1+n^2} + \left\{\begin{matrix}\tan^{-1}\dfrac{2n}{1-n^2} \\ \text{or} \\ \sin^{-1}\dfrac{2n}{1+n^2}\end{matrix}\right\}\right] \tag{7}$$

This formula checks that given by Terzaghi, "Erdbaumechanik," p. 226, or "Report of Committee on Earths and Foundations," Proc. A.S.C.E., May, 1933, p. 786. As n approaches zero, σ_z becomes $\frac{4nw}{\pi}$ or $\frac{2p}{\pi Z}$ where p is the load per unit length. This checks the form of Boussinesq's equation for a line load, $\sigma_z = \frac{2p}{\pi}\frac{z^3}{R^4}$, when $R = z$, or for points vertically under the load.

5. *Table of Vertical Pressures.*—By using Equation (5), values of $\frac{\sigma_z}{w}$ at the corner of a uniformly loaded rectangle for a number of values of m and n have been computed,* and are given in Table 1. In the table, m and n are interchangeable.

*The form most convenient for computations will depend on the tables of trignometric functions available. The writer used "Sieben- und mehrstellige Tafeln der Kreis- und Hyperbelfunktionen," by K. Hayashi, Julius Springer, Berlin, 1926. There are a number of typographical errors in this book, but by using both the $\sin^{-1}$ and $\tan^{-1}$ forms, the values can be checked.

6. *Numerical Examples.*—Examples of the use of the table follow.

Example I

("Report of Committee on Earths and Foundations," p. 784.) Compute the intensity of vertical stress at points 25 ft. directly below (a) the center and (b) the corners of a raft foundation, 20 ft. wide and 60 ft. long, carrying a uniform load of 3 tons per sq. ft.

(a) For vertical stress under the center, combine four equal rectangles of dimensions 10 by 30 ft.

$$m = \frac{10}{25} = 0.4, \; n = \frac{30}{25} = 1.2$$

From Table 1, the coefficient = 0.10631,

$$\text{whence } \sigma_z = 4\,(0.10631) \times 3 = 1.276 \text{ tons per sq. ft.}$$

(The result obtained by Professor Gilboy by means of dividing the area into small elements is given in the report as 1.31 tons per sq. ft.).

(b) For vertical stress under corner,

$$m = \frac{20}{25} = 0.8, \; n = \frac{60}{25} = 2.4$$

By interpolating between values of $n = 2.0$ and $n = 2.5$, the coefficient obtained is 0.18281,

$$\text{whence } \sigma_z = 0.18281 \times 3 = 0.548 \text{ tons per sq. ft.}$$

(The result given in the report is 0.55 tons per sq. ft.)

Example II

Find the vertical stress 40 ft. below the corner of a loaded area 44 ft. by 34 ft., when $w = 2$ tons per sq. ft.

$$m = \frac{44}{40} = 1.1, \; n = \frac{34}{40} = 0.85$$

It is necessary to interpolate twice to obtain the desired coefficient. Interpolating for $m = 1.1$ between $m = 1.0$ and 1.2, for $n = 0.8$, we get 0.16411, and for $n = 0.9$, 0.17301. Interpolating between these values for $n = 0.85$, we get 0.16856; whence the stress is $0.16856 \times 2 = 0.337$ tons per sq. ft.

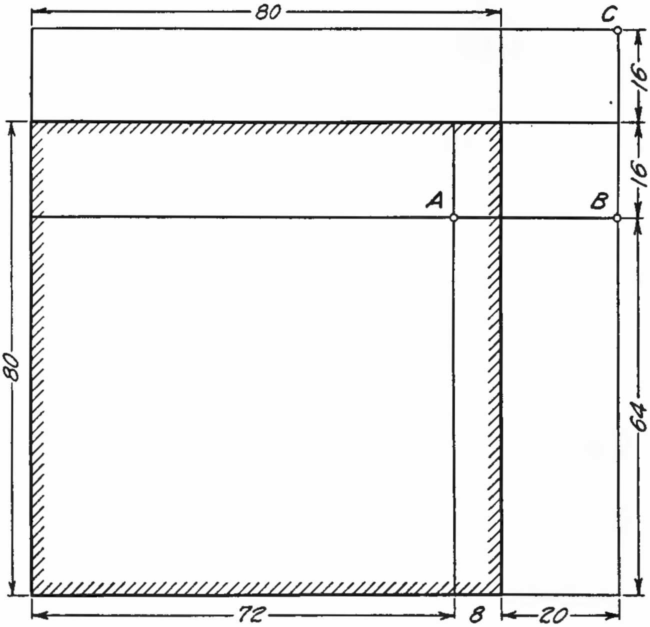

FIG. 3. LOADED AREA USED FOR NUMERICAL EXAMPLE

Example III

Given a load of 1 ton per sq. ft. on an area 80 ft. by 80 ft., as in Fig. 3, find the vertical stress at points A, B, C, at a distance of 40 ft. below the surface.

Point A. The sum due to four areas, $\sigma_z = w \cdot \sum f_{mn}$, as follows:

$$f_{0.2,\,0.4} + f_{0.2,\,1.6} + f_{1.8,\,0.4} + f_{1.8,\,1.6}$$
$$= 0.03280 + 0.05994 + 0.11260 + 0.22372 = 0.42906$$
$$\sigma_z = 0.42906 \text{ tons per sq. ft.}$$

Point B.

$$f_{2.5,\,1.6} + f_{2.5,\,0.4} - f_{0.5,\,1.6} - f_{0.5,\,0.4}$$
$$= 0.22940 + 0.11450 - 0.13241 - 0.07111 = 0.14038$$
$$\sigma_z = 0.14038 \text{ tons per sq. ft.}$$

Point C.

$$f_{2.4,\,2.5} - f_{2.4,\,0.5} - f_{0.4,\,2.5} + f_{0.4,\,0.5}$$
$$= 0.23931 - 0.13602 - 0.11450 + 0.07111 = 0.05990$$
$$\sigma_z = 0.05990 \text{ tons per sq. ft.}$$

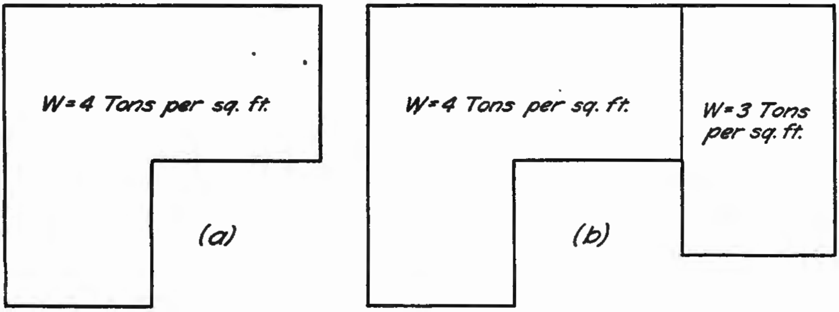

FIG. 4. LOADED AREAS TO WHICH METHOD OF COMPUTATION IS APPLICABLE

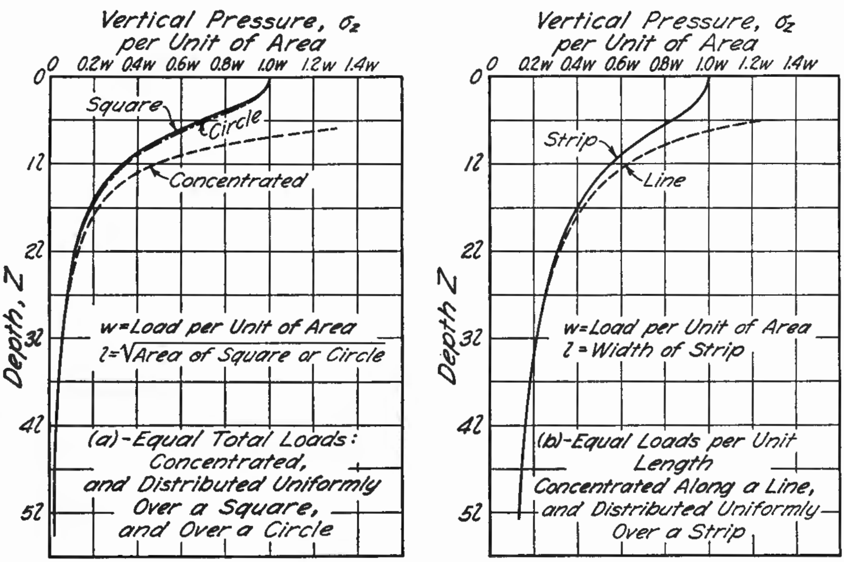

FIG. 5. VERTICAL STRESS AT POINTS UNDER RESULTANT OF LOAD

Applications to more complicated plans of loading, as in Fig. 4, follow directly along the same lines.

7. *Error Involved in Assuming Distributed Loads as Concentrated.*— It is of some interest to compare the vertical stress at different depths beneath the center of several plans of loading. Fig. 5 (a) shows the variation in vertical stress with depth below equal total loads, concentrated and distributed uniformly over a square and over a circle; Fig. 5 (b) shows the variation in vertical stress with depth below

equal loads per unit length, concentrated, and distributed uniformly over a strip.

From these diagrams it appears that at depths which are large relative to the size of the loaded area it makes no difference what the distribution of load is; the pressure is practically the same as for a concentrated load. Near the surface, the stress for a concentrated load approaches infinite values, and the stress for a uniform load approaches the value of the load intensity.

It is possible to deduce expressions for the error involved by considering a distributed load as concentrated. First, consider the case of the vertical stress at a depth Z below the center of a circle of radius R, loaded with intensity w. Integration of Boussinesq's formula for this case yields*

$$\sigma_z = w\left[1 - \left\{\frac{1}{1+\left(\frac{R}{Z}\right)^2}\right\}^{3/2}\right] \tag{8}$$

The total load is $\pi R^2 w$, and the stress considering the load as concentrated is

$$\sigma_z' = \frac{3}{2}\,\frac{\pi R^2 w}{\pi}\cdot\frac{1}{Z^2} = w\cdot\frac{3}{2}\left(\frac{R}{Z}\right)^2 \tag{9}$$

Expanding Equation (8) by the binomial theorem, for small values of $\frac{R}{Z}$,

$$\sigma_z = w\left[1 - \left\{1 - \frac{3}{2}\left(\frac{R}{Z}\right)^2 + \frac{15}{8}\left(\frac{R}{Z}\right)^4 - \frac{105}{48}\left(\frac{R}{Z}\right)^6 + \ldots\right\}\right] =$$

$$\frac{3}{2}w\left(\frac{R}{Z}\right)^2\left[1 - \frac{5}{4}\left(\frac{R}{Z}\right)^2 + \frac{35}{24}\left(\frac{R}{Z}\right)^4 - \ldots\right]$$

The error, e,

$$= \sigma_z' - \sigma_z = \frac{3}{2}w\left(\frac{R}{Z}\right)^2\left[\frac{5}{4}\left(\frac{R}{Z}\right)^2 - \frac{35}{24}\left(\frac{R}{Z}\right)^4 + \ldots\right]$$

and the relative error, $\frac{e}{\sigma_z}$,

*For example, see Föppl, "Drang und Zwang," Vol. II, p. 230.

$$= \frac{5}{4}\left(\frac{R}{Z}\right)^2 \frac{1 - \frac{7}{6}\left(\frac{R}{Z}\right)^2 + \ldots}{1 - \frac{5}{4}\left(\frac{R}{Z}\right)^2 + \frac{35}{24}\left(\frac{R}{Z}\right)^4 - \ldots}$$

and, to a fairly close approximation,

$$\frac{e}{\sigma_z} = \sim\frac{5}{4}\left(\frac{R}{Z}\right)^2 \tag{10}$$

For $\frac{R}{Z} = 1.0$ the relative error by actual computation is 1.320, and by Equation (10) is 1.250. For smaller values of $\frac{R}{Z}$ Equation (10) is much more accurate. For example, if $\frac{R}{Z} = 0.5$, the true relative error is 0.3183, and the value given by the approximate formula is 0.3125.

Thus the error involved in the assumption that a load distributed uniformly over a circle of diameter $D = 2R$ is concentrated at the center of the circle, is, for $D = \frac{1}{2}Z$, about 7.82 per cent; for $D = \frac{1}{3}Z$, about 3.47 per cent, and for $D = \frac{1}{4}Z$, about 1.95 per cent.

By a similar process, and by comparing true and approximate values of the relative error for a square area of side S, the value is found to be, approximately,

$$\frac{e}{\sigma_z} = \sim\frac{10}{24}\left(\frac{S}{Z}\right)^2 \tag{11}$$

For $S = 2Z$, by actual computation the relative error is 1.73, and that given by Equation (11), 1.67. For $S = Z$, the true and approximate values are respectively 0.420 and 0.417.

Thus the error involved in assuming that a load distributed uniformly over a square of side S is concentrated at the center, is, for $S = \frac{1}{2}Z$, about 10.4 per cent; for $S = \frac{1}{3}Z$, about 4.63 per cent; and for $S = \frac{1}{4}Z$, about 2.60 per cent.

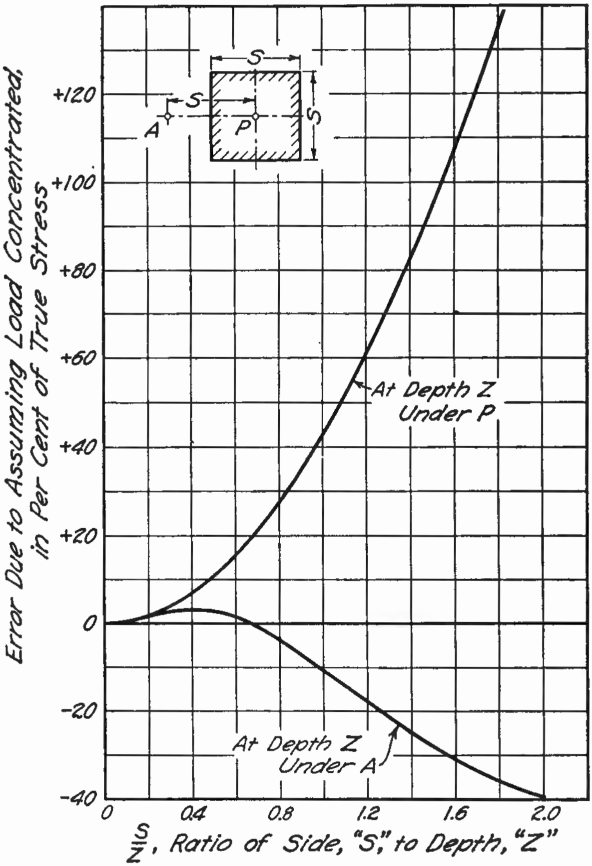

FIG. 6. ERROR IN TERMS OF TRUE VALUE OF STRESS DUE TO ASSUMING UNIFORMLY DISTRIBUTED LOAD AS CONCENTRATED AT CENTER OF SQUARE LOADED AREA

If the loaded area is divided up into a number of squares, or into approximately square areas, the relative error will be less than the values just given, since the relative error is smaller for areas not directly over the point where the stress is computed, and may even become negative. For example, the relative error in stress under points A and P in Fig. 6 due to assuming the load on the square area of side S as concentrated at P is given in the figure plotted against the ratio of $\frac{S}{Z}$, where Z is the depth at which the stress is computed under points A and P.

Quoting from the "Report of the Committee on Earths and Foundations," p. 783: "In considering the distribution of pressures below a footing of average size the question arises as to the proper subdivision of the footing into units of such size that the load on each unit is sufficiently small to be considered a point concentration as assumed in the Boussinesq formula. Professor Gilboy has made a thorough investigation of this point, and he concludes that a division of the area into rectangular elements the longest side of which is less than one-half the distance from the element to the point (at which the stress is to be computed) will give a correct result to within 6%; less than one-third the distance, correct to within 3%; and less than one-fourth the distance, correct to within 2%."

These rules are approximately correct when the footing is of such size that there will be several elements to be considered. However, when the loaded area is of just such size that only one concentration is to be considered, the rules will be in error. But considering the uncertainties involved in computing pressure and settlement in soils, this error is probably insignificant.

8. *Concluding Remarks.*—Since Boussinesq's formula is used for lack of a better way of determining vertical pressures in foundations, the writer feels that any simplification in the procedure used in applying the formula is of some value. The table given herein permits a rapid and simple calculation of the pressures, and is presented for that reason.

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